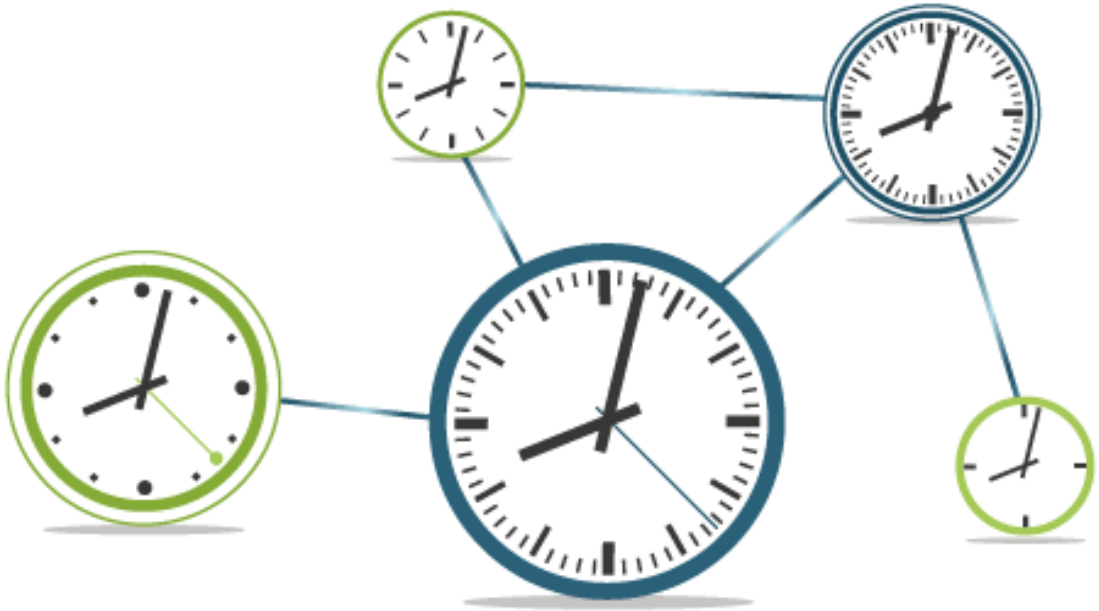
Synchronization



- Synchronization: A phenomenon in which multiple oscillators vibrate homogeneously through interaction
- A system where clocks spinning at different speeds adjust their speed by looking at the clock connected to them?
- Synchronization of fireflies, synchronization of applause at concert halls, etc.

Kuramoto model

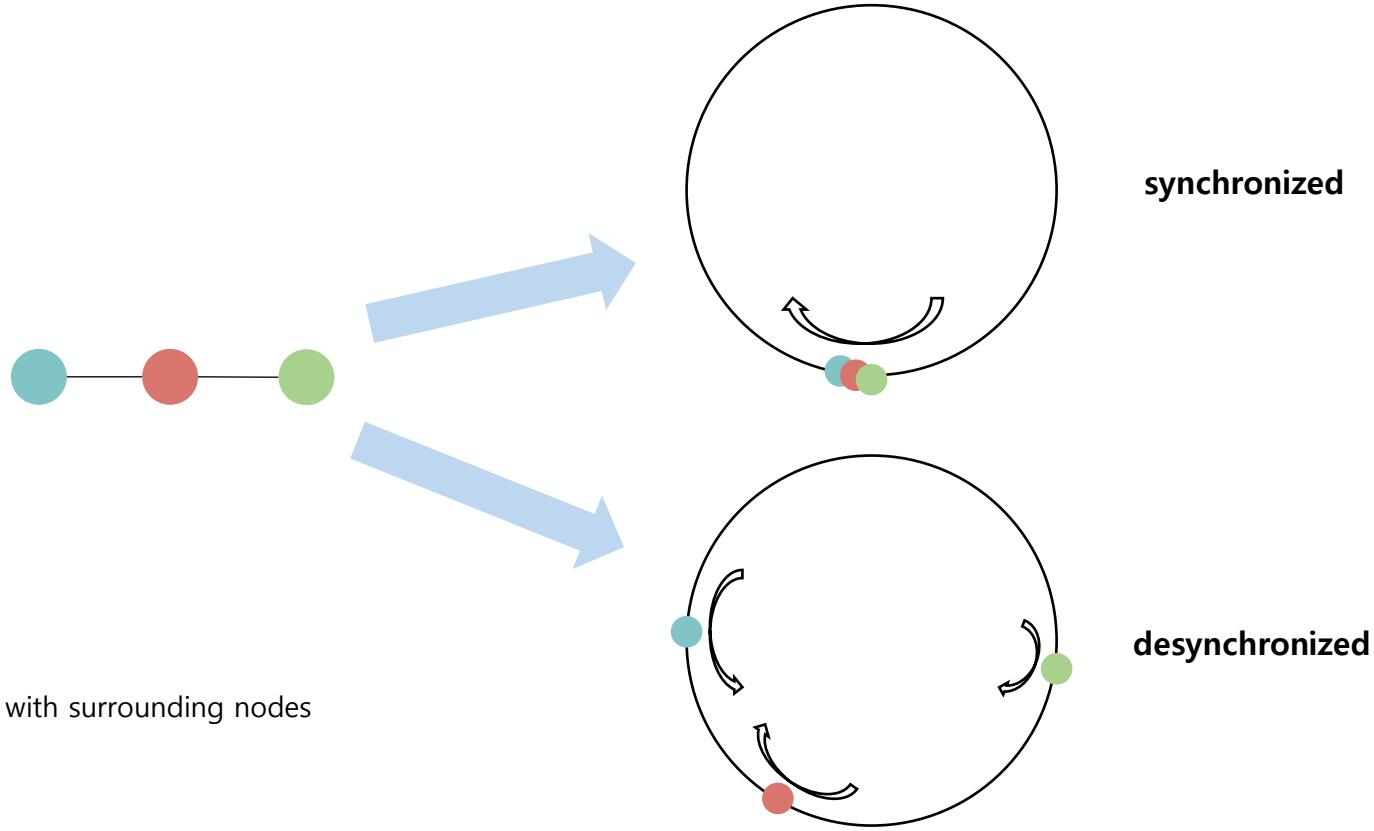
Angular velocity of node i

Interaction intensity

$$\frac{d\theta_i}{dt} = \Omega_i + \sum_j K_{ij} sin(\theta_j - \theta_i)$$

natural frequency

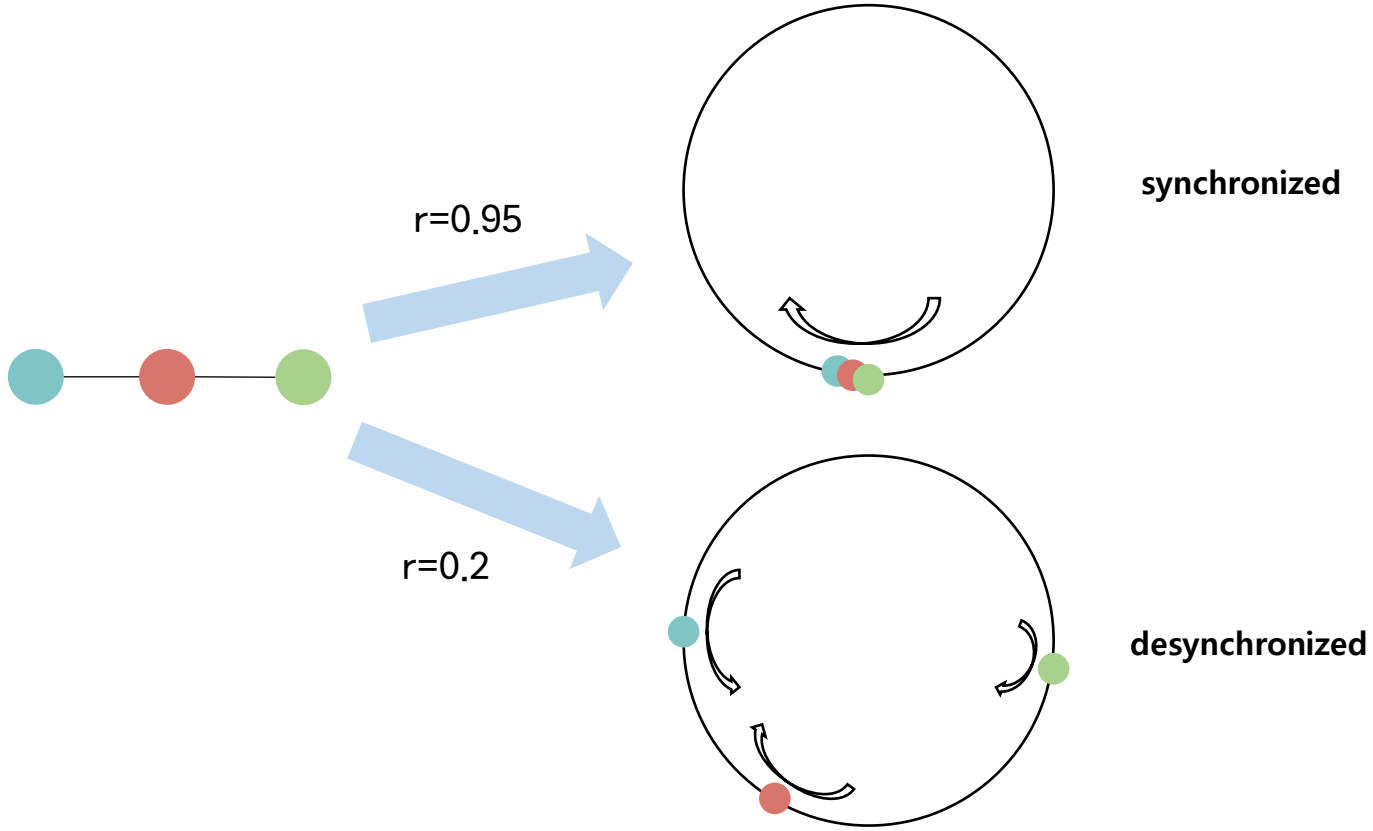
Interaction with surrounding nodes



- Kuramoto model: the simplest model for synchronization
- A state where everyone looks at each other and applauds.
- Depending on the interaction strength (K), the system can be synchronized or desynchronized.

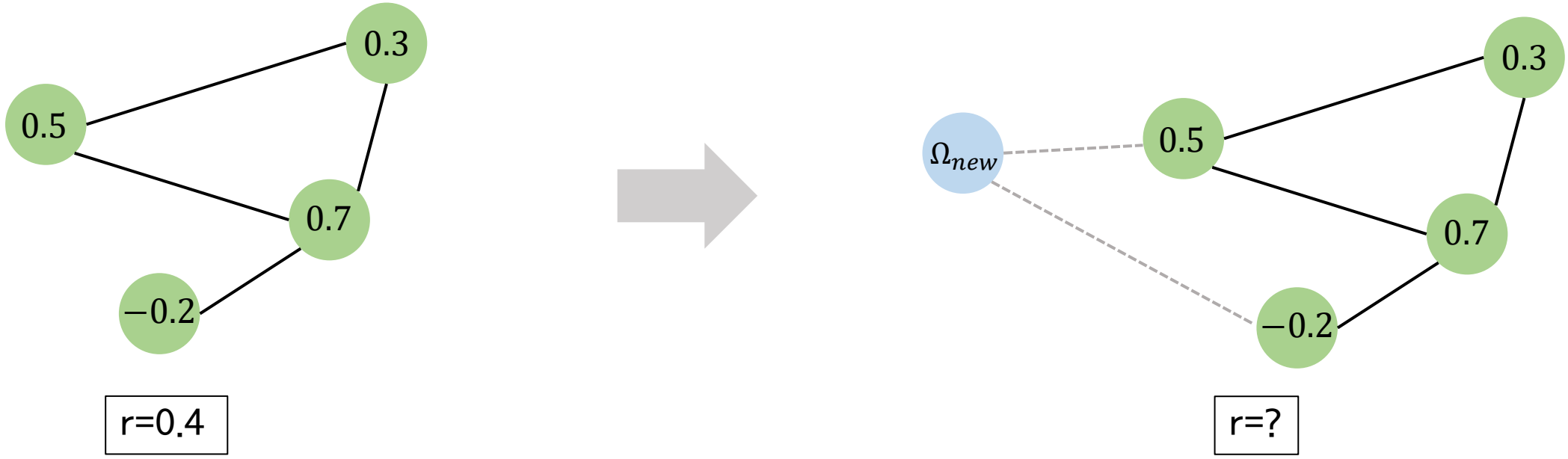
Kuramoto model

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$



- Order parameter: A variable that expresses how well the synchronization is (0~1)
- The larger r, the better the synchronization.

Optimal frequency for growing network

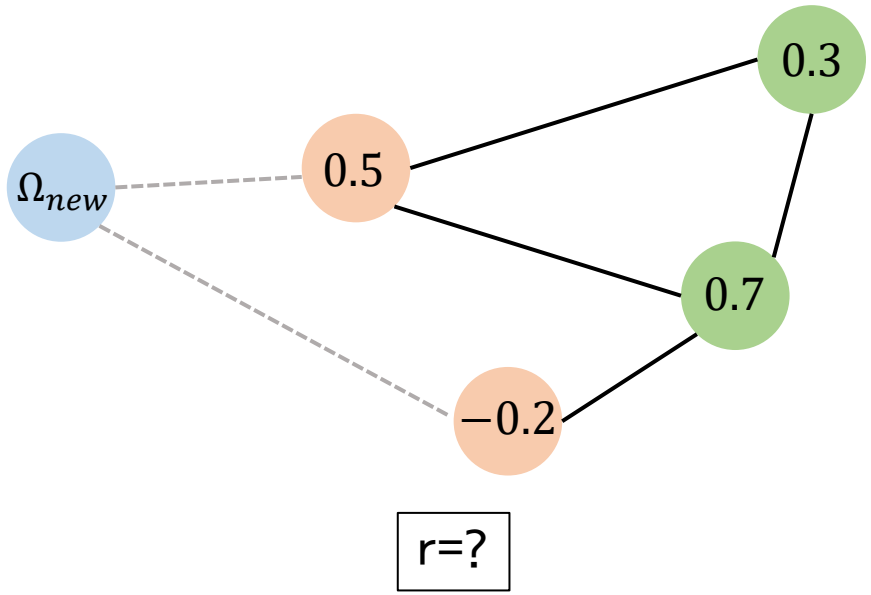


$\text{Network}(K_{ij})$
$\{\Omega_i\}$
→ Algorithm → Ω_{new}

- Question: When adding a new node to a network, how should its natural frequency be assigned?
- Compute the frequency to preserve or improve the original network's order parameter r.
- An algorithm that infers the new node's frequency from the existing network and its frequency distribution.

Optimal frequency for growing network

- Global: Average of all existing node frequencies
- Local: Average value of adjacent nodes (red)
- Random: Just random (normal distribution)
- SAF
- ALF



- Grow a given network (30 nodes) to 40 and assign frequencies to new nodes in 5 ways.
- Comparison of r at the beginning and r at the end
- What is the best way to increase the r of the network?

Optimal frequency for growing network

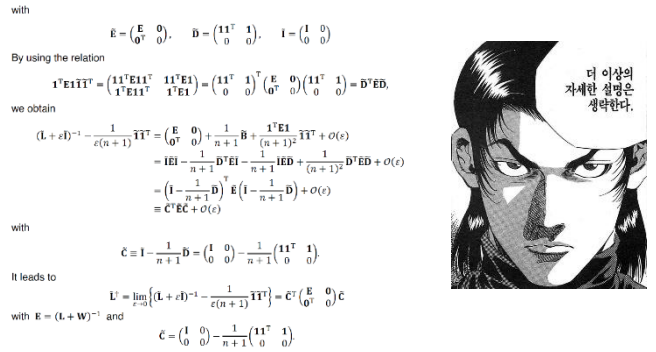
SAF: I want to increase the order parameter as much as possible.

$$\omega_{N+1}^{*} = \underset{\omega_{N+1}}{\operatorname{argmin}} \left\{ \boldsymbol{\omega}^{\mathrm{T}} \left(\mathbf{L}^{\dagger} \right)^{2} \boldsymbol{\omega} \right\}$$

ALF: We want to reduce the power loss of the entire network as much as possible.

$$\omega_{N+1}^{*} = \underset{\omega_{N+1}}{\operatorname{argmin}} \left\{ \boldsymbol{\omega}^{\mathrm{T}} \mathbf{L}^{\dagger} \boldsymbol{\omega} \right\}$$

induction:


with

$$\bar{\mathbf{E}} = \begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}, \quad \bar{\mathbf{D}} = \begin{pmatrix} \mathbf{1}\mathbf{1}^{\mathrm{T}} & \mathbf{1} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}, \quad \bar{\mathbf{I}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}$$

By using the relation

$$\mathbf{1}^{\mathrm{T}}\mathbf{E}\mathbf{1}\bar{\mathbf{1}}\bar{\mathbf{1}}^{\mathrm{T}} = \begin{pmatrix} \mathbf{1}\mathbf{1}^{\mathrm{T}}\mathbf{E}\mathbf{1}\mathbf{1}^{\mathrm{T}} & \mathbf{1}\mathbf{1}^{\mathrm{T}}\mathbf{E}\mathbf{1} \\ \mathbf{1}^{\mathrm{T}}\mathbf{E}\mathbf{1}\mathbf{1}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}}\mathbf{E}\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1}\mathbf{1}^{\mathrm{T}} & \mathbf{1} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}\mathbf{1}^{\mathrm{T}} & \mathbf{1} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix} = \bar{\mathbf{D}}^{\mathrm{T}}\bar{\mathbf{E}}\bar{\mathbf{D}},$$

we obtain

$$\begin{aligned} (\mathbf{L} + \varepsilon\bar{\mathbf{I}})^{-1} - \frac{1}{\varepsilon(n+1)}\bar{\mathbf{1}}\bar{\mathbf{1}}^{\mathrm{T}} &= \begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix} + \frac{1}{n+1}\bar{\mathbf{B}} + \frac{\mathbf{1}^{\mathrm{T}}\mathbf{E}\mathbf{1}}{(n+1)^2}\bar{\mathbf{1}}\bar{\mathbf{1}}^{\mathrm{T}} + \mathcal{O}(\varepsilon) \\ &= \bar{\mathbf{I}}\bar{\mathbf{E}}\bar{\mathbf{I}} - \frac{1}{n+1}\bar{\mathbf{D}}^{\mathrm{T}}\bar{\mathbf{E}}\bar{\mathbf{I}} - \frac{1}{n+1}\bar{\mathbf{I}}\bar{\mathbf{E}}\bar{\mathbf{D}} + \frac{1}{(n+1)^2}\bar{\mathbf{D}}^{\mathrm{T}}\bar{\mathbf{E}}\bar{\mathbf{D}} + \mathcal{O}(\varepsilon) \\ &= \left(\bar{\mathbf{I}} - \frac{1}{n+1}\bar{\mathbf{D}}\right)^{\mathrm{T}}\bar{\mathbf{E}}\left(\bar{\mathbf{I}} - \frac{1}{n+1}\bar{\mathbf{D}}\right) + \mathcal{O}(\varepsilon) \\ &\equiv \bar{\mathbf{C}}^{\mathrm{T}}\bar{\mathbf{E}}\bar{\mathbf{C}} + \mathcal{O}(\varepsilon) \end{aligned}$$

with

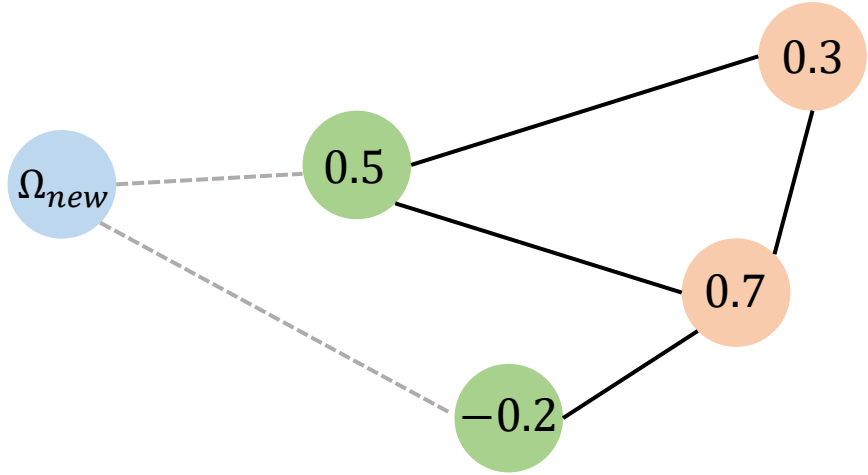
$$\bar{\mathbf{C}} \equiv \bar{\mathbf{I}} - \frac{1}{n+1}\bar{\mathbf{D}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix} - \frac{1}{n+1}\begin{pmatrix} \mathbf{1}\mathbf{1}^{\mathrm{T}} & \mathbf{1} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}.$$

It leads to

$$\bar{\mathbf{L}}^{\dagger} = \lim_{\varepsilon \to 0}\left\{(\mathbf{L} + \varepsilon\bar{\mathbf{I}})^{-1} - \frac{1}{\varepsilon(n+1)}\bar{\mathbf{1}}\bar{\mathbf{1}}^{\mathrm{T}}\right\} = \bar{\mathbf{C}}^{\mathrm{T}}\begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}\bar{\mathbf{C}}$$

with $\mathbf{E} = (\mathbf{L} + \mathbf{W})^{-1}$ and

$$\bar{\mathbf{C}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix} - \frac{1}{n+1}\begin{pmatrix} \mathbf{1}\mathbf{1}^{\mathrm{T}} & \mathbf{1} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}$$



- SAF and ALF compute the natural frequency of a new node in a growing network based on established results.
- SAF: approximates the frequency that maximizes the order parameter.
- ALF: approximates the frequency that minimizes mutual interactions (≈power loss)

SAF method [P. S. Skardal, D. Taylor, and J. Sun, PRL **113**, 144101 (2014)]

Strong coupling regime ($r \approx 1$; $|\theta_i(t)| \ll 1$)

Equation of motion

$$\dot{\theta}_i(t) = \omega_i - K\sum_{i=1}^{N} A_{ij} \sin\left(\theta_i(t) - \theta_j(t)\right)$$

$$\rightarrow \dot{\theta}_i(t) = \omega_i - K\sum_{i=1}^{N} L_{ij}\theta_j(t)$$

$L_{ij} = \delta_{ij}k_i - A_{ij}$: Laplace matrix

$k_i = \sum_j A_{ij}$: degree of node i

Order parameter

$$r^* e^{i\psi^*} = \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_i^*}$$

$$\rightarrow r^* = 1 - \frac{1}{2N}\sum_{i=1}^{N} (\theta_i^*)^2$$

$$\dot{\boldsymbol{\theta}}(t) = \boldsymbol{\omega} + K\mathbf{L}\boldsymbol{\theta}(t), \qquad r^* = 1 - \frac{1}{2N}(\boldsymbol{\theta}^*)^{\mathrm{T}}\boldsymbol{\theta}^*$$

SAF method [P. S. Skardal, D. Taylor, and J. Sun, PRL **113**, 144101 (2014)]

Strong coupling regime ($r \approx 1$; $|\theta_i(t)| \ll 1$)

Equation of motion

$$\dot{\boldsymbol{\theta}}(t) = \boldsymbol{\omega} - K\mathbf{L}\boldsymbol{\theta}(t) \rightarrow \boldsymbol{\theta}^* = \frac{1}{K}\mathbf{L}^{\dagger}\boldsymbol{\omega}$$

Order parameter

$$r^* = 1 - \frac{1}{2N}(\boldsymbol{\theta}^*)^{\mathrm{T}}\boldsymbol{\theta}^*$$

$\mathbf{L}^{\dagger} = \sum_{i=2}^{N} \lambda_i^{-1}\boldsymbol{\nu}_i\boldsymbol{\nu}_i^{\mathrm{T}}$: Moore-Penrose inverse

$$\rightarrow r^* = 1 - \frac{1}{2K^2N}\boldsymbol{\omega}^{\mathrm{T}}\left(\mathbf{L}^{\dagger}\right)^2\boldsymbol{\omega}$$

r^* is larger $\leftrightarrow$ $\boldsymbol{\omega}^{\mathrm{T}}\left(\mathbf{L}^{\dagger}\right)^2\boldsymbol{\omega}$ is smaller

Measure for the optimization

$$\omega_{N+1}^* = \underset{\omega_{N+1}}{\operatorname{argmin}}\left\{\boldsymbol{\omega}^{\mathrm{T}}\left(\mathbf{L}^{\dagger}\right)^2\boldsymbol{\omega}\right\}$$

ALF method [Y. Lei, *et. al.*, to be published]

Strong coupling regime ($r \approx 1$; $|\theta_i(t)| \ll 1$)

Equation of motion

$$\dot{\boldsymbol{\theta}}(t) = \boldsymbol{\omega} - K\mathbf{L}\boldsymbol{\theta}(t) = \boldsymbol{\omega} - \boldsymbol{\nabla}_{\boldsymbol{\theta}} V\big(\boldsymbol{\theta}(t)\big) \rightarrow \boldsymbol{\theta}^* = \frac{1}{K}\mathbf{L}^{\dagger}\boldsymbol{\omega}$$

$V(\boldsymbol{\theta}) = \frac{1}{2} K \boldsymbol{\theta}^{\mathrm{T}} \mathbf{L} \boldsymbol{\theta}$: power loss (effective potential)

$$V(\boldsymbol{\theta}^*) = \frac{1}{2K}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{L}^{\dagger}\mathbf{L}\mathbf{L}^{\dagger}\boldsymbol{\omega} = \frac{1}{2K}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{L}^{\dagger}\boldsymbol{\omega} \qquad (\mathbf{L}^{\dagger}\mathbf{L}\mathbf{L}^{\dagger} = \mathbf{L}^{\dagger})$$

smaller $V(\boldsymbol{\theta}^*)$ would be better (?)

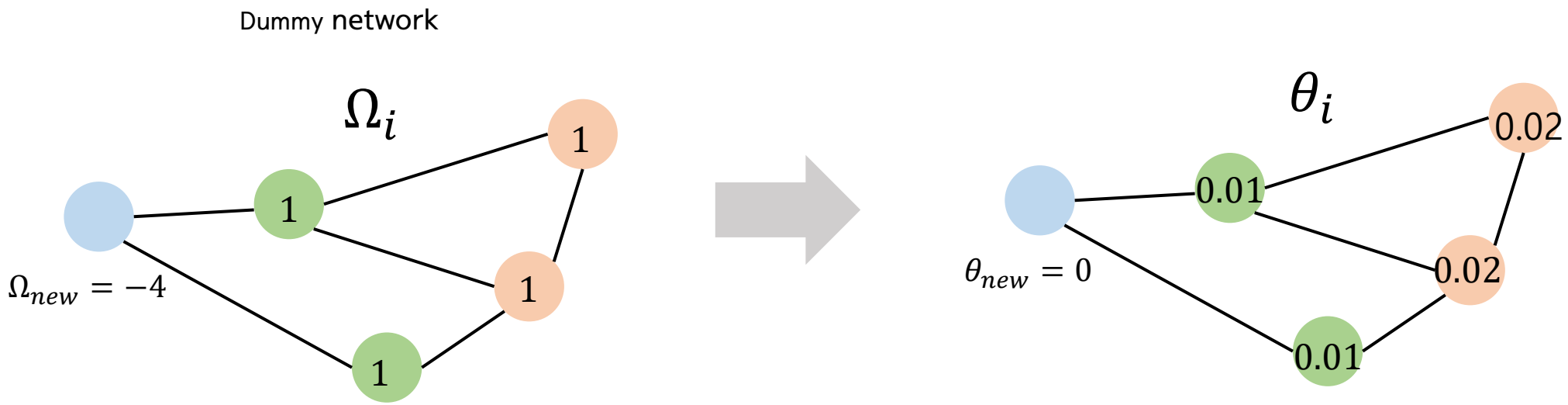
Measure for the optimization

$$\omega_{N+1}^* = \underset{\omega_{N+1}}{\operatorname{argmin}}\left\{\boldsymbol{\omega}^{\mathrm{T}}\mathbf{L}^{\dagger}\boldsymbol{\omega}\right\}$$

Optimal frequency for growing network

ALF: We want to reduce the power loss of the entire network as much as possible.

$$\omega_{N+1}^{*} = \underset{\omega_{N+1}}{\operatorname{argmin}}\{\boldsymbol{\omega}^{\mathrm{T}}\mathbf{L}^{\dagger}\boldsymbol{\omega}\}$$



- Another interpretation of ALF: construct a dummy network where all existing nodes are assigned an angular velocity of 1.
- In its synchronized state, a larger phase difference with the new node leads to stronger influence on its assigned frequency.
- The new node's angular velocity is determined by weighting green nodes by 1 and red nodes by 2.

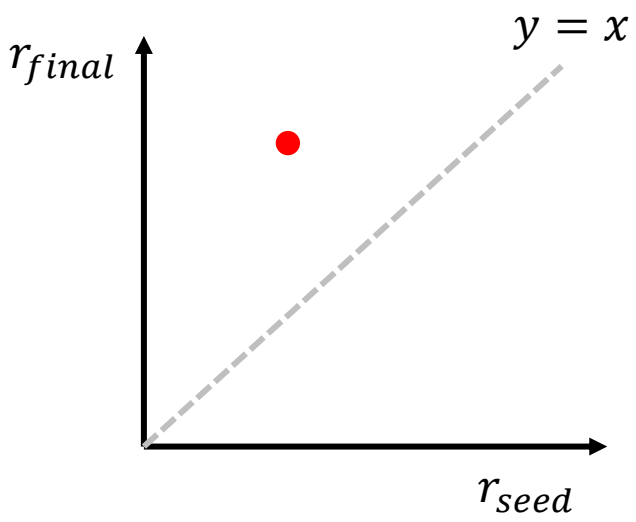
Optimal frequency for growing network

$G(50,p)$	Ω_1	Ω_2	$\cdots$	Ω_{50}	
$net\ 1$					r_1
$net\ 2$					r_2
		$\mathbb{N}(0,1)$			
$\vdots$					$\vdots$
$net\ 300$					r_{300}
					r_{seed}

- Test method: Create several seed networks using the same settings
- Generating the natural frequency of each node as a normal distribution.
- The average value of r obtained from 300 networks becomes the representative value of the seed network.

Optimal frequency for growing network

$G(50,p)$	Ω_1	Ω_2	$\cdots$	Ω_{50}		$\Omega_{51}\sim\Omega_{60}$	
$net\ 1$					r_1		r_1
$net\ 2$					r_2		r_2
		$\mathbb{N}(0,1)$				$Local$	
$\vdots$					$\vdots$		$\vdots$
$net\ 300$					r_{300}		r_{300}
					r_{seed}		r_{final}

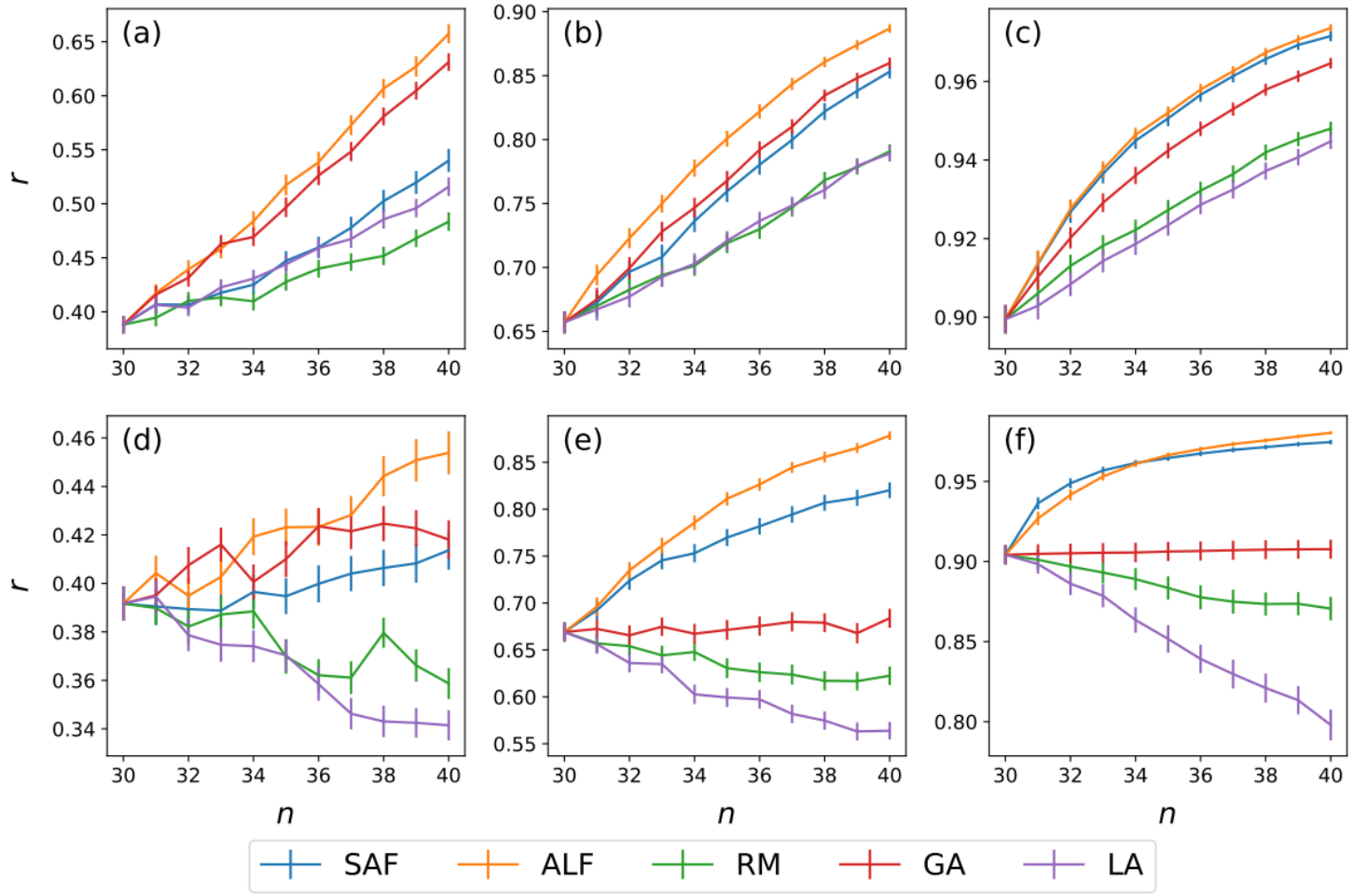


- Test method: Create several seed networks using the same settings
- Generating the natural frequency of each node as a normal distribution.
- The average value of r obtained from 300 networks becomes the representative value of the seed network.

Result

Callaway

BA

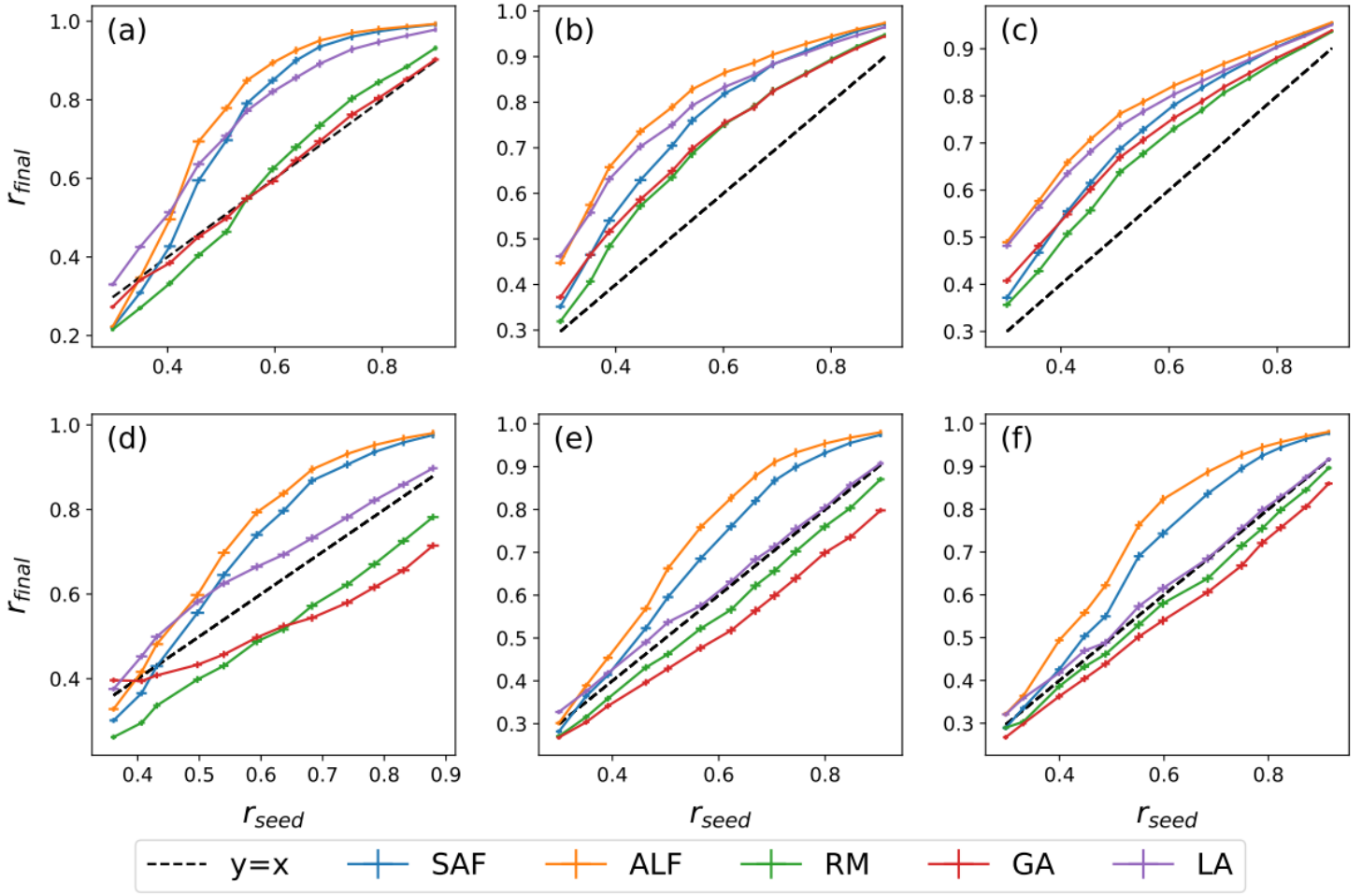


- Comparison of order parameters while growing 10 more seed networks with N=30
- When each column starts from a low r region, a middle region, or a high region
- ALF is good in most areas, SAF is a bit ambiguous

Optimal frequency for growing network

Callaway

BA



- Difference between r of seed network and final r according to network size
- In rows 1, 2, and 3, the network sizes are 10~20, 30~40, and 50~60.
- ALF is good except for small networks.