COMPLEX POWER SYSTEM RESEARCH GROUP

Optimal Link addition in Kuramoto Model for Enhanced synchronization

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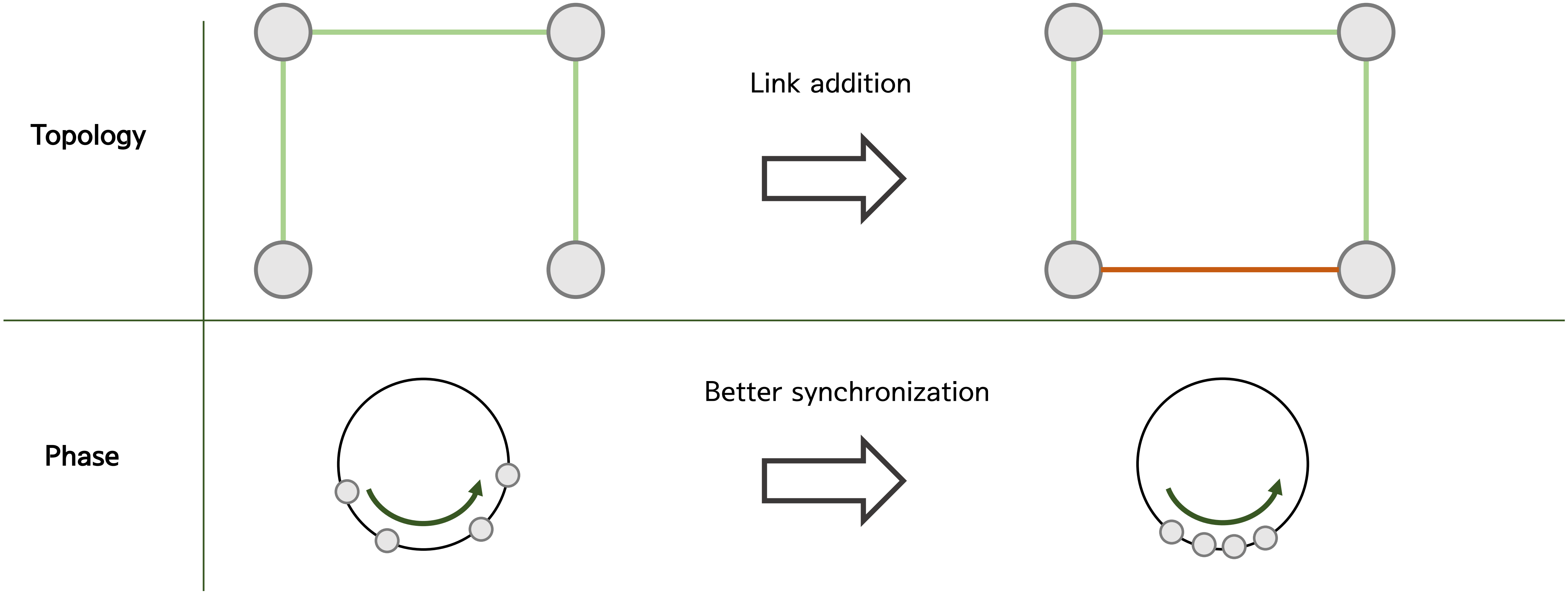
Link addition in oscillatory system



- Adding new links to the oscillator network generally promotes synchronization.
- The degree of synchronization increases or decreases depending on the location of the new link.
- How to locate new links on the target network to maximize synchronization?

First-order Kuramoto model

Angular velocity

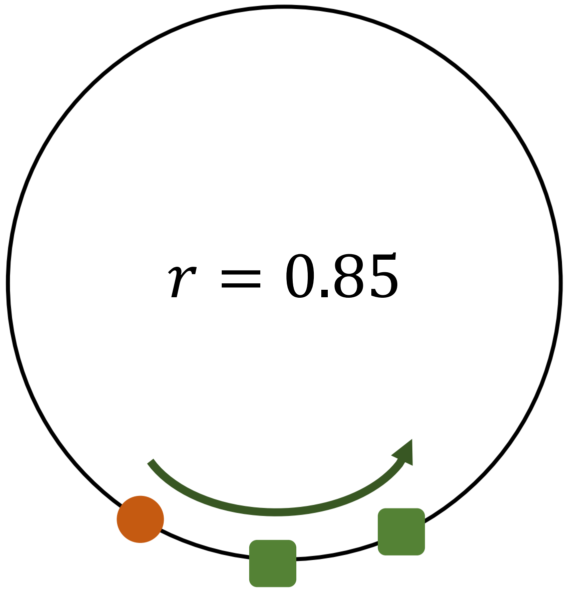
Coupling constant

$$\frac{d\theta_i}{dt} = \omega_i + K \sum_j A_{ij} sin(\theta_j - \theta_i)$$

Natural frequency

Interaction

Order parameter

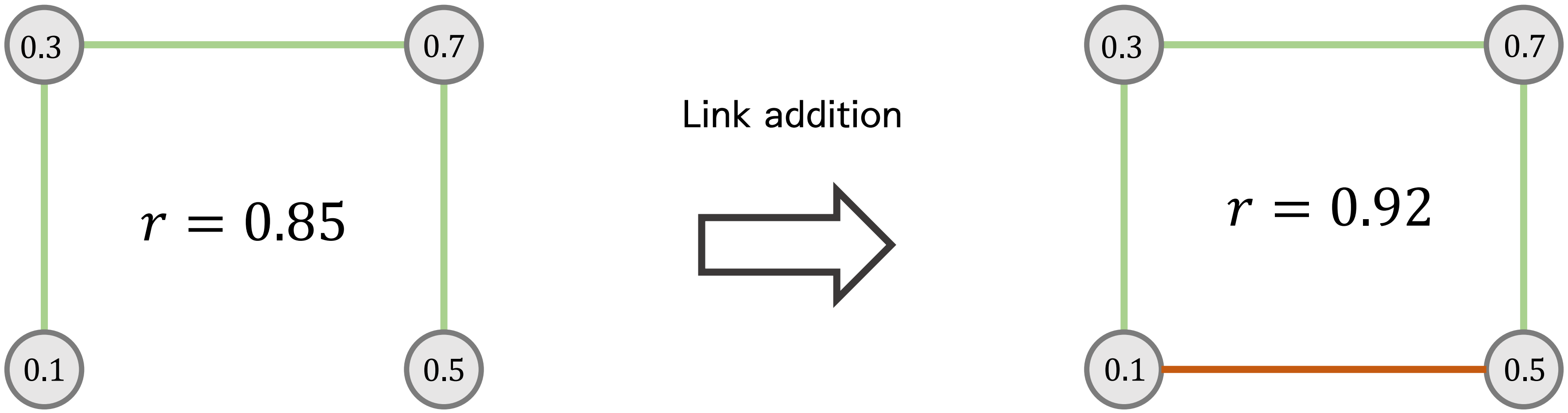


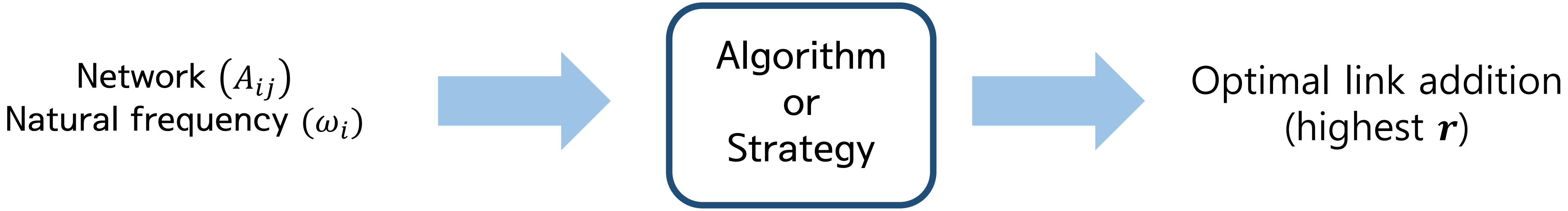
$$r = \frac{1}{N} \left| \sum_i e^{i\theta_i} \right|$$

Number of nodes

- First-order Kuramoto model: The simplest model to describe the motion of an inertial oscillator.
- Order parameter: Index measuring the degree of synchronization of the system (0~1)

Stready state of the Kuramoto model





- Goal: find optimal link additions using the network structure and natural frequencies
- Metric: final order parameter $(\boldsymbol{r})$

Stready state of the Kuramoto model

$$\frac{d\theta_i}{dt} = \omega_i + K\sum_j A_{ij} sin(\theta_j - \theta_i)$$

Strong coupling regime

$\sin(\theta) \approx \theta$

$$\frac{d\theta_i}{dt} = \omega_i + K\sum_j \boxed{A_{ij}(\theta_j - \theta_i)}$$

Laplace matrix

$L_{ij} = \delta_{ij}k_i - A_{ij}$

k_i =i's degree

$$\frac{d\theta_i}{dt} = \omega_i + K\sum_j \boxed{L_{ij}\theta_j}$$

- SAF method[1)]: Find a function that corresponds to the network input and the order parameter of the system.
- When the interaction is sufficiently strong, the Kuramoto model can be linearized and expressed in matrix form.

1) Lei et al. (2023), Chaos, Solitons & Fractals, 168.

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Steady state $\left(\frac{\boldsymbol{d\theta_i}}{\boldsymbol{dt}} = \boldsymbol{0}\right)$

$$0 = \boldsymbol{\omega} + KL\boldsymbol{\theta}$$

Moore-Penrose inverse

$$L^{\dagger} = \sum_{i=2} \lambda_i^{-1} v_i v_i^T$$

$$\boxed{\boldsymbol{\theta}^* = \frac{1}{K} L^{\dagger} \boldsymbol{\omega}}$$

Phase vector (output)

Frequency vector (input)

Network structure (input)

- SAF method[1]: Find a function that corresponds to the network input and the order parameter of the system.
- When the interaction is sufficiently strong, the Kuramoto model can be linearized and expressed in matrix form.
- The phase vector of the system can be obtained from the input.

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Synchrony Alignment Function (SAF)

$$\boxed{r} = \frac{1}{N}\left|\sum_i e^{i\theta_i}\right|$$

$$= \frac{1}{N}\left|\sum_i 1 + \cancel{i\theta_i} - \frac{\theta_i^2}{2} + \cdots\right|$$

$$\approx 1 - \frac{1}{2N}\sum \theta_i^2$$

$$= \boxed{1 - \frac{1}{2N}\boldsymbol{\theta}^T\boldsymbol{\theta}}$$

- The order parameter of the system can also be approximated by a determinant in the strong coupling regime.

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$$r = 1 - \frac{1}{2NK^2}\boldsymbol{\omega}\left(L^\dagger\right)^2\boldsymbol{\omega}$$

Order parameter
(output)

Network structure
(input)

Frequency vector
(input)

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- Summary by substituting steady state →Approximate expression for order parameter in steady state

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$$\Downarrow$$

$$\operatorname{argmax}(r)$$

$$\boxed{= \operatorname{argmin}\left(\boldsymbol{\omega}\left(L^\dagger\right)^2\boldsymbol{\omega}\right)}$$

- The order parameter of the system can also be approximated by a determinant in the strong coupling regime.
- Summary by substituting steady state →Approximate expression for order parameter in steady state
- Link to maximizing the order parameter of the system = Problem of finding the minimum point of the SAF object function

Adjusted Lyapnov Function (ALF)

$$\boxed{r_{loc}} = K \sum_{i,j} \cos(\theta_i - \theta_j)$$

$$= K \sum_{i,j} 1 - \frac{(\theta_i - \theta_j)^2}{2} + \cdots$$

$$\approx KN - \frac{K}{2} \sum (\theta_i - \theta_j)^2$$

$$= \boxed{KN - \frac{K\boldsymbol{\theta}^T L \boldsymbol{\theta}}{2}}$$

- ALF function[2)] : Similar to SAF, but uses local order parameter (total interaction energy)
- Similar to SAF, but an object function whose order of Laplace matrix is 1

2) Skardal et al. (2014), Phys. Rev. Lett., 113(14), 144101.

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$$r_{loc} = KN - \frac{1}{2K} \boldsymbol{\omega} L^{\dagger} \boldsymbol{\omega}$$

Local Order parameter
(output)

Network structure
(input)

Frequency vector
(input)

$$\boldsymbol{\theta}^* = \frac{1}{K} L^{\dagger} \boldsymbol{\omega}$$

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$$= KN - \frac{K \boldsymbol{\theta}^{\boldsymbol{T}} L \boldsymbol{\theta}}{2}$$

$$r_{loc} = KN - \frac{1}{2K} \boldsymbol{\omega} L^{\dagger} \boldsymbol{\omega}$$

$$\Downarrow$$

$$\text{argmax}(r_{loc})$$

$$\boxed{= \text{argmin}\left(\boldsymbol{\omega} L^{\dagger} \boldsymbol{\omega}\right)}$$

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Prediction of synchrony enhancement

SAF	$\operatorname{argmin}\left(\boldsymbol{\omega}\left(L^{\dagger}\right)^{\mathbf{2}}\boldsymbol{\omega}\right)$	$\underset{dL}{\operatorname{argmin}}\left(\boldsymbol{\omega}\left((L+dL)^{\dagger}\right)^{\mathbf{2}}\boldsymbol{\omega}\right)$	$\operatorname{argmax}\left(\dfrac{2\boldsymbol{\theta}^{T}L^{\dagger}dL\boldsymbol{\theta}}{1+X^{T}L^{\dagger}X}-\dfrac{\boldsymbol{\theta}^{T}dL\left(L^{\dagger}\right)^{2}dL\boldsymbol{\theta}}{(1+X^{T}L^{\dagger}X)^{2}}\right)$
ALF	$\operatorname{argmin}\left(\boldsymbol{\omega}L^{\dagger}\boldsymbol{\omega}\right)$	$\underset{dL}{\operatorname{argmin}}\left(\boldsymbol{\omega}(L+dL)^{\dagger}\boldsymbol{\omega}\right)$	$\underset{i,j}{\operatorname{argmax}}\left(\dfrac{\left(\theta_i^{*}-\theta_j^{*}\right)^{2}}{1+X^{T}L^{\dagger}X}\right)$

- As links are added to the network, the Laplace matrix changes. (dL)
- The problem of finding a dL matrix that optimizes two types of object functions
- The matrix sum is organized using the formula for Moore-Penrose inverse.

$$(L+dL)^{\dagger}=(L+X^{T}X)^{\dagger}=\left(L^{+}-\frac{L^{+}XX^{T}L^{+}}{(1+X^{T}L^{+}X)}\right)$$

$$dL=X^{T}X$$

$$X_k=\delta_{ik}-\delta_{jk}$$

$$Ex)\ i=2, j=4$$

$$X=\begin{pmatrix}0 & 1 & 0 & -1 & 0\end{pmatrix}$$

Prediction of synchrony enhancement

SAF	$\operatorname{argmin}\left(\boldsymbol{\omega}\left(L^{\dagger}\right)^{\mathbf{2}}\boldsymbol{\omega}\right)$	$\underset{dL}{\operatorname{argmin}}\left(\boldsymbol{\omega}\left((L+dL)^{\dagger}\right)^{\mathbf{2}}\boldsymbol{\omega}\right)$	$\operatorname{argmax}\left(\frac{2\boldsymbol{\theta}^T L^{\dagger} dL\boldsymbol{\theta}}{1+X^T L^{\dagger} X}-\frac{\boldsymbol{\theta}^T dL\left(L^{\dagger}\right)^2 dL\boldsymbol{\theta}}{(1+X^T L^{\dagger} X)^2}\right)$
ALF	$\operatorname{argmin}\left(\boldsymbol{\omega}L^{\dagger}\boldsymbol{\omega}\right)$	$\underset{dL}{\operatorname{argmin}}\left(\boldsymbol{\omega}(L+dL)^{\dagger}\boldsymbol{\omega}\right)$	distant nodes in phase space $\underset{i,j}{\operatorname{argmax}}\left(\frac{\left(\theta_i^*-\theta_j^*\right)^2}{1+X^T L^{\dagger} X}\right)$

$$(L+dL)^{\dagger}=(L+X^TX)^{\dagger}=\left(L^{+}-\frac{L^{+}XX^TL^{+}}{(1+X^TL^{+}X)}\right)$$

$$dL=X^TX$$

$$X_k=\delta_{ik}-\delta_{jk}$$

$$Ex)\ i=2, j=4$$

$$X=(0\quad 1\quad 0\quad -1\quad 0)$$

- ALF's object function can be interpreted in a more intuitive sense.
- Numerator: Connect nodes where the phase difference between steady states is as large as possible.

Prediction of synchrony enhancement

SAF	$\mathrm{argmin}\left(\boldsymbol{\omega}\left(L^{\dagger}\right)^{\mathbf{2}}\boldsymbol{\omega}\right)$	$\underset{dL}{\mathrm{argmin}}\left(\boldsymbol{\omega}\left((L+dL)^{\dagger}\right)^{\mathbf{2}}\boldsymbol{\omega}\right)$	$\mathrm{argmax}\left(\frac{2\boldsymbol{\theta}^{T}L^{\dagger}dL\boldsymbol{\theta}}{1+X^{T}L^{\dagger}X}-\frac{\boldsymbol{\theta}^{T}dL\left(L^{\dagger}\right)^{2}dL\boldsymbol{\theta}}{(1+X^{T}L^{\dagger}X)^{2}}\right)$
ALF	$\mathrm{argmin}\left(\boldsymbol{\omega}L^{\dagger}\boldsymbol{\omega}\right)$	$\underset{dL}{\mathrm{argmin}}\left(\boldsymbol{\omega}(L+dL)^{\dagger}\boldsymbol{\omega}\right)$	$\underset{i,j}{\mathrm{argmax}}\left(\frac{\left(\theta_i^{*}-\theta_j^{*}\right)^{2}}{1+X^{T}L^{\dagger}X}\right)$ close nodes in network structure

$$(L+dL)^{\dagger}=(L+X^{T}X)^{\dagger}=\left(L^{+}-\frac{L^{+}XX^{T}L^{+}}{(1+X^{T}L^{+}X)}\right)$$

$$dL=X^{T}X \qquad X_k=\delta_{ik}-\delta_{jk}$$

$$Ex)\ i=2, j=4$$

$$X=(0\quad 1\quad 0\quad -1\quad 0)$$

- ALF's object function can be interpreted in a more intuitive sense.
- Numerator: Connect nodes where the phase difference between steady states is as large as possible.
- Denominator: Connect nodes that are as close together as possible in the net work structure.

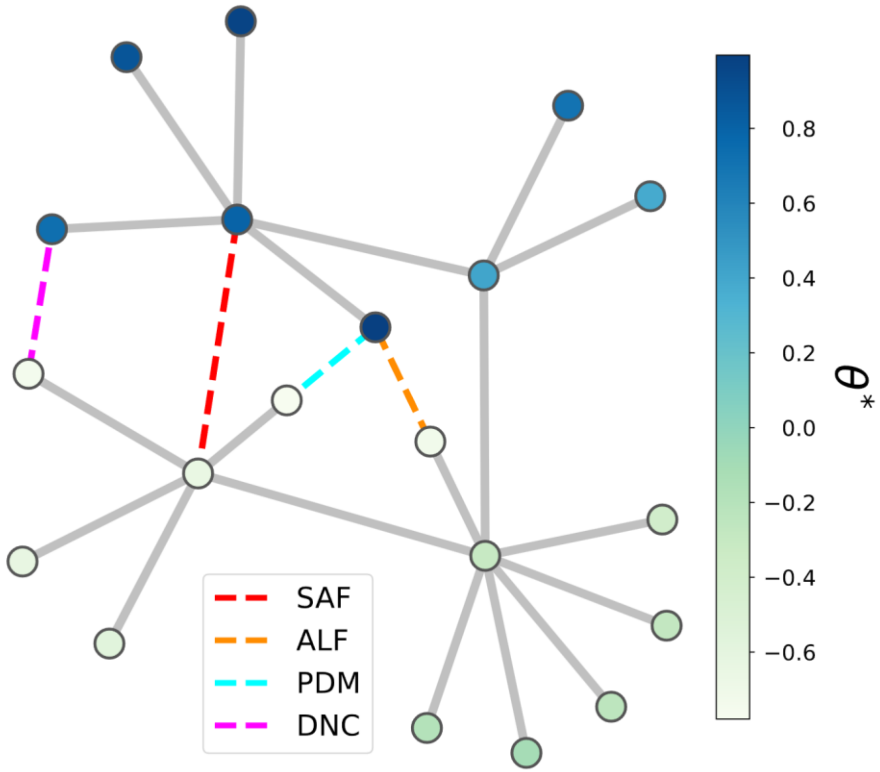
Prediction of synchrony enhancement

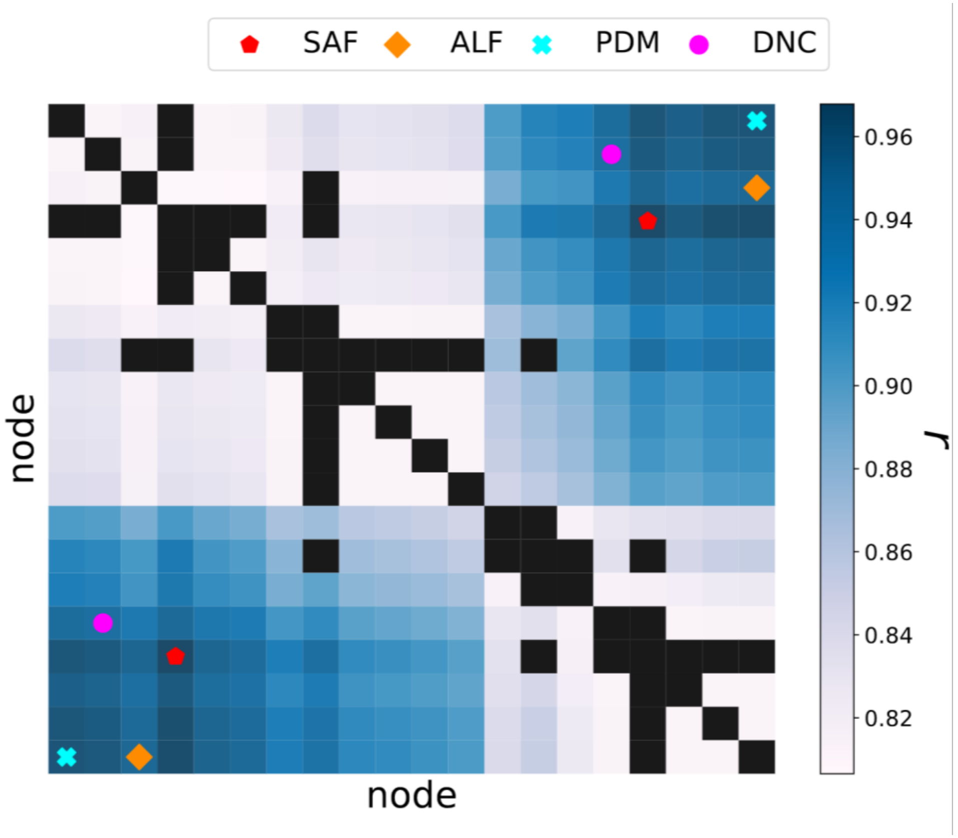
SAF	$\frac{2\boldsymbol{\theta}^T L^\dagger dL\boldsymbol{\theta}}{1+X^T L^\dagger X} - \frac{\boldsymbol{\theta}^T dL\left(L^\dagger\right)^2 dL\boldsymbol{\theta}}{(1+X^T L^\dagger X)^2}$	= Node pair that maximizes the order parameter
PDM	$\left(\theta_i^* - \theta_j^*\right)^2$	= Node pair with the largest phase difference
DNC	$X^T L^\dagger X$	= Node pair with the greatest structural distance
ALF	$\frac{\left(\theta_i^* - \theta_j^*\right)^2}{1+X^T L^\dagger X}$	= Node pairs that maximize the local order parameter = Pair with large phase difference but close distance

- Separate the two terms to create a separate link addition strategy
- Four strategies for adding links based on different aspects of your network

Prediction of synchrony enhancement

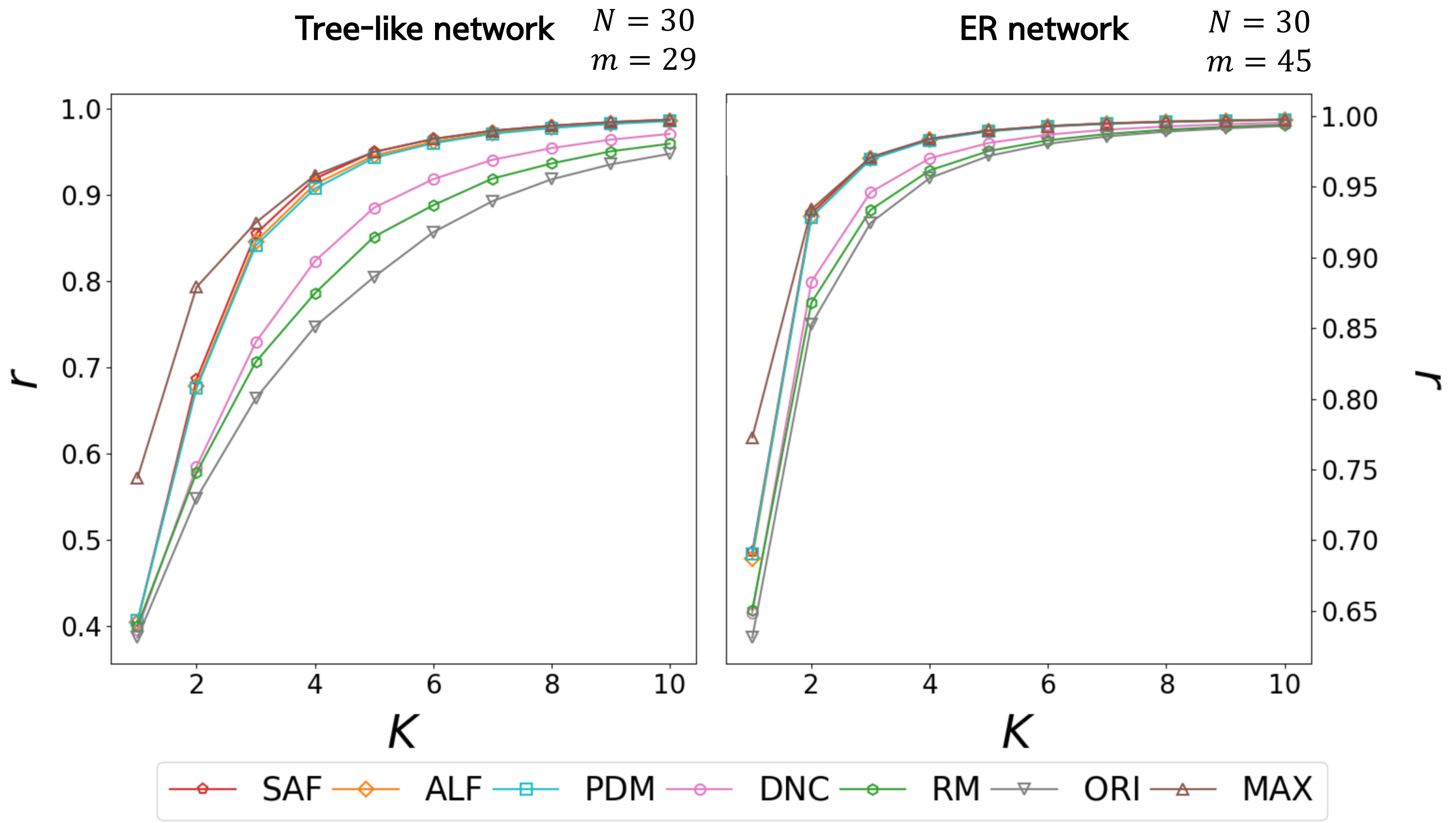
$N = 20$
$m = 19$
$K = 5$





- Analysis of connection trends of each method in model network
- DNC connects the farthest location
- PDM connects the location with the largest phase difference
- ALF connects the location with the largest phase difference but closer.
- SAF, which directly maximizes the order parameter, shows the best performance.

Prediction of synchrony enhancement



- Performance comparison of four methods under various K and network structures
- RM: random connection, ORI: original network, MAX: performance of optimal link obtained by brute force.
- SAF is the best in all intervals, and is close to the correct answer when K is large.

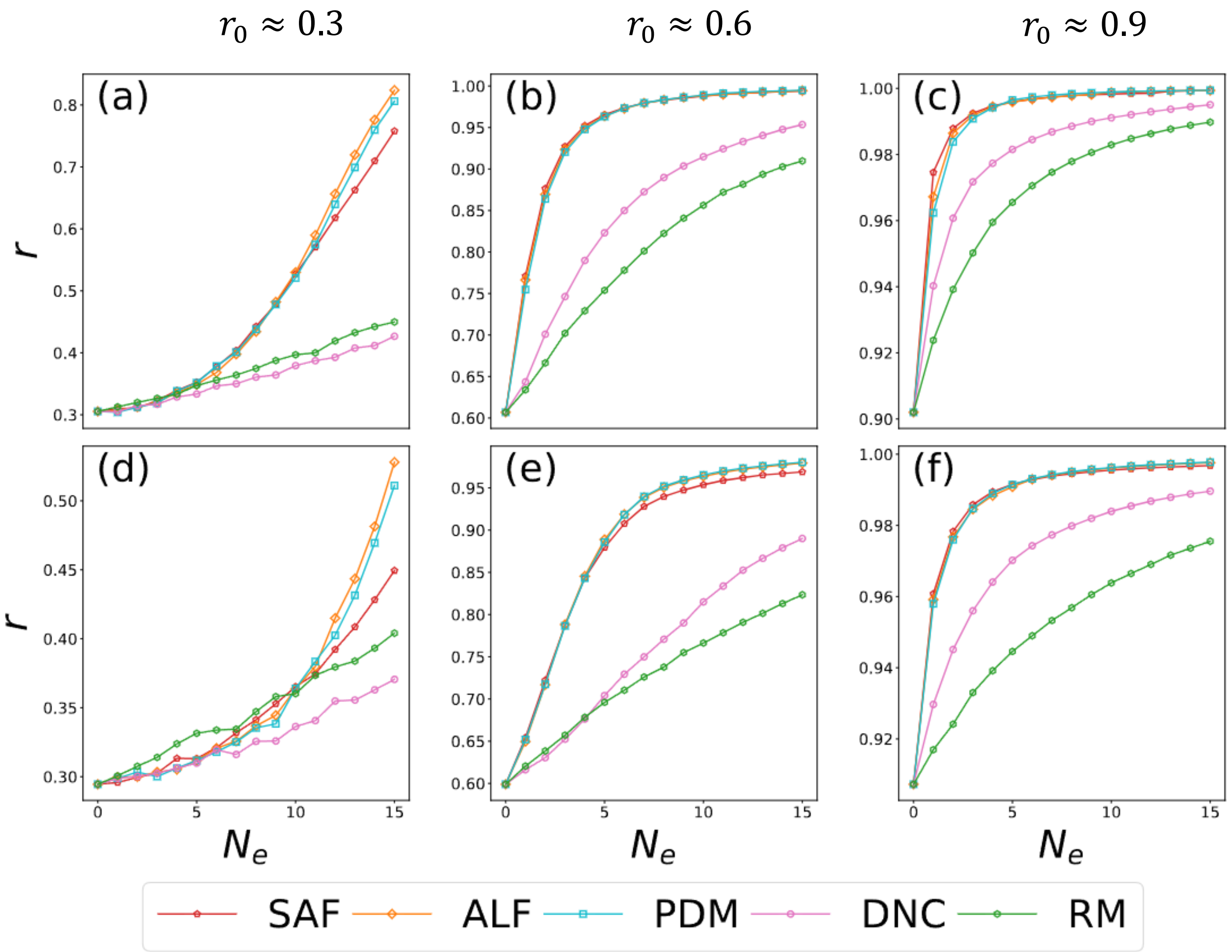
Prediction of synchrony enhancement

Tree-like network

$N = 30$
$m = 29$

ER network

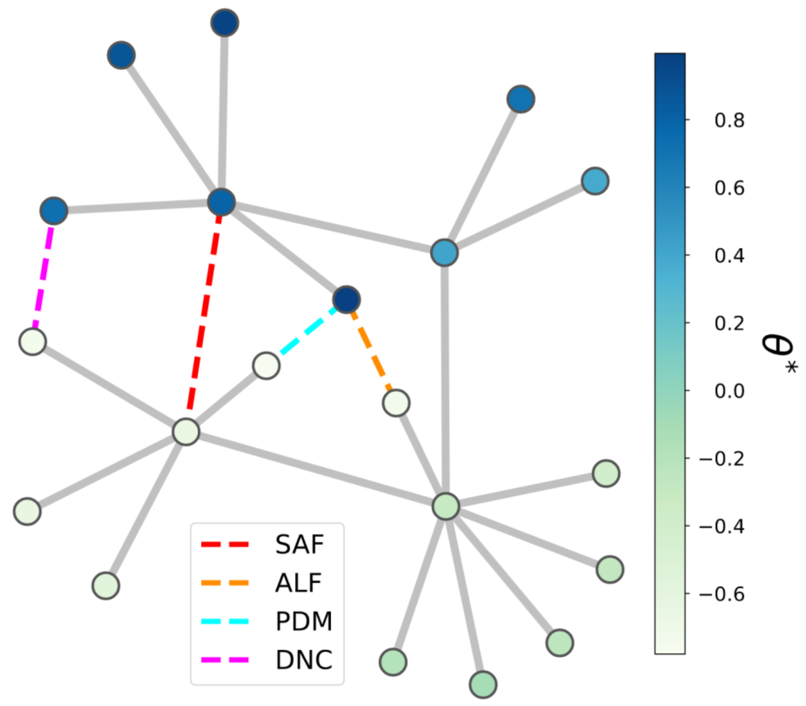
$N = 30$
$m = 45$



- Observing the trends with more links
- SAF is still the best in most sections.
- When K is low or the number of added links increases, ALF and PDM show better performance.

Summary

- Searching for links that most improve synchronization in oscillator networks
- Using strategies that maximize different objective functions
- SAF, which optimizes the order parameter, performs best
- Reflecting both the topology and structural information of the network



Future work

- What is the optimal strategy when two or more links are added?
- Applied to optimization problems such as power grid systems